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Strength distribution and gauge length extrapolations in glass fibre

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Tensile strength data of E and S-994 glass fibres, reported by previous workers, have been analysed using a modified Weibull distribution. The function provides an upper and a lower limit of strength and is characterized by two shape parameters. Based on the strength data at two gauge lengths (15 and 120 mm), predictions of the strengths at other gauge lengths are in good agreement with experimental data.

1. Introduction

Glass fibres are brittle materials and as such they exhibit a very broad tensile strength distribution. The fibre strength is determined by pre-existing surface flaws [1] which occur with a statistical nature. This increases the probability of encountering a more severe flaw with increase in surface area, with the consequent reduction of the fibre strength as the length increases. Statistically, the dependence of fibre strength on its length is analysed in terms of weakest-link theory and the form most frequently applied to reinforcing fibres such as glass or carbon is due to Weibull [2]. Current theories of composite strength [3–6] usually require a knowledge of the fibre strength and its distribution at gauge lengths of the order of the critical length for the stress transfer (0.1 to 0.2 mm) and the Weibull distribution forms the basis of extrapolating the fibre strength to such gauge lengths. Practical limitations impose restrictions for obtaining reliable data from experimental measurements at such extremely short gauge lengths.

Schmitz and Metcalfe [7] studied the effect of length on the strength of glass fibres of two different compositions (E-glass and S-994 glass) using gauge lengths in the range 0.25 to 240 mm. They analysed the experimental data in terms of Gaussian and Weibull distributions and the Kies [8] modification of the Weibull distribution. As opposed to the prediction of these theories, the log(strength)–log(length) plot showed a change in slope at short fibre lengths (below 5 mm). This was explained in terms of a bimodal flaw distribution and the need for a more reliable strength–length model was felt. In this paper a modification to the Weibull distribution is suggested for the analysis of fibre strength data. The applicability of the suggested function has also been evaluated in terms of the experimental values reported by Schmitz and Metcalfe [7].

2. Cumulative flaw distribution function

The "weakest link" theory is based on the assumption that a brittle material fails when the stress at any one flaw becomes larger than the ability of the surrounding material to resist the local stresses. Thus the failure probability P, that a fibre of length L will fail below stress level S, is given by [9]

$$P = 1 - \exp\left[-L\,N(S)\right] \qquad (1)$$

where $N(S)$ is the cumulative number of flaws of strength less than S per unit length.

In Equation 1, $N(S)$ is an unknown function and Weibull assumed an empirical form for this function, given by

$$\begin{aligned} N(S) &= \left(\frac{S - S_L}{S_0}\right)^m \qquad S > S_L \\ N(S) &= 0 \qquad\qquad\qquad S < S_L \end{aligned} \qquad (2)$$

where S_L is the stress at which there is zero probability of failure, S_0 is a normalizing factor and m is a shape parameter, usually referred as the Weibull modulus. In most fracture work S_L is taken as zero since it is a location parameter only and does not change the goodness of fit of data to the distribution function [10]. Thus Equations 1 and 2 yield

$$\ln\left[\frac{1}{L}\ln\left(\frac{1}{1-P}\right)\right] = m\ln S - m\ln S_0 \qquad (3)$$

Equation 3 shows that a plot of $\ln\{(1/L)\ln[1/(1-P)]\}$ against $\ln S$ for the strength data tested at the same or different gauge lengths should yield a straight line with slope m. However, for numerous strength data reported in the literature [7, 9, 10–15] for glass fibres, carbon fibres as well as glass rods, the data points do not fall on a single straight line as predicted from Equation 3 and it has been concluded [12] that the Weibull distribution is not applicable to the entire strength distribution. As an alternative approach a bimodal Weibull distribution has usually been applied [9, 12] for the analysis of the data, with two separate straight lines fitted to the data according to Equation 3. It may be pointed out that the strict application of the Weibull distribution presumes a unimodal distribution [16]. Olshansky and Maurer [9] have pointed out that a bimodal strength distribution in which the slope of $\ln[N(S)]$ decreases with $\ln S$ cannot be interpreted in terms of two Weibull

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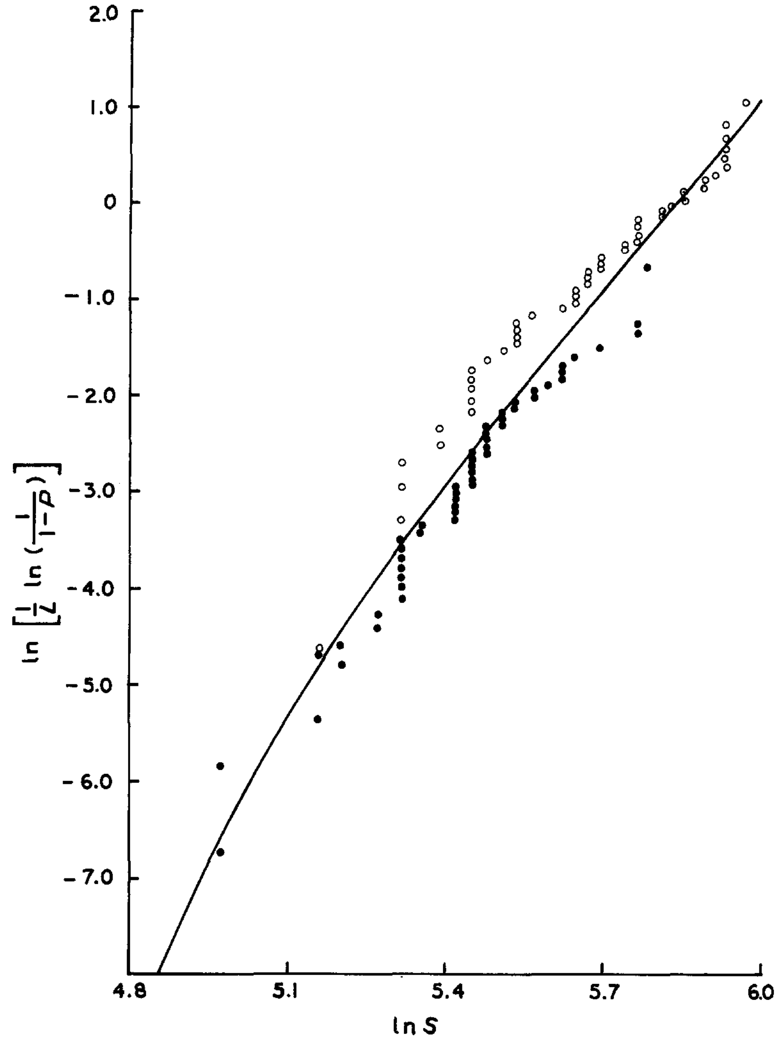

Figure 1 Plot of ln $\{(1/L) \ln [1/(1 - P)]\}$ against ln S for E-glass fibres at gauge lengths of (○) 15 and (●) 120 mm, from Smitz and Metcalfe [7]. The curve corresponds to the fitted equation.

distributions unless it is assumed that neither distribution extends over the entire experimental range of stresses. Such a data plot is shown in Fig. 1. It shows the tensile strength data for E-glass fibre reported by Schmitz and Metcalfe [7] for gauge lengths of 15 and 120 nm. As can be seen from the plot, the flaw population seems to change its character at about 208.5×10^3 p.s.i. (1.44 GPa). The histograms of tensile strength data for these two gauge lengths, as shown in Figs 2a and b, indicate unimodal distributions rather than a non-overlapping bimodal distribution. An analysis of these two distributions in terms of the Pearson system of probability-density functions [17] yields unimodal beta distributions, which are also

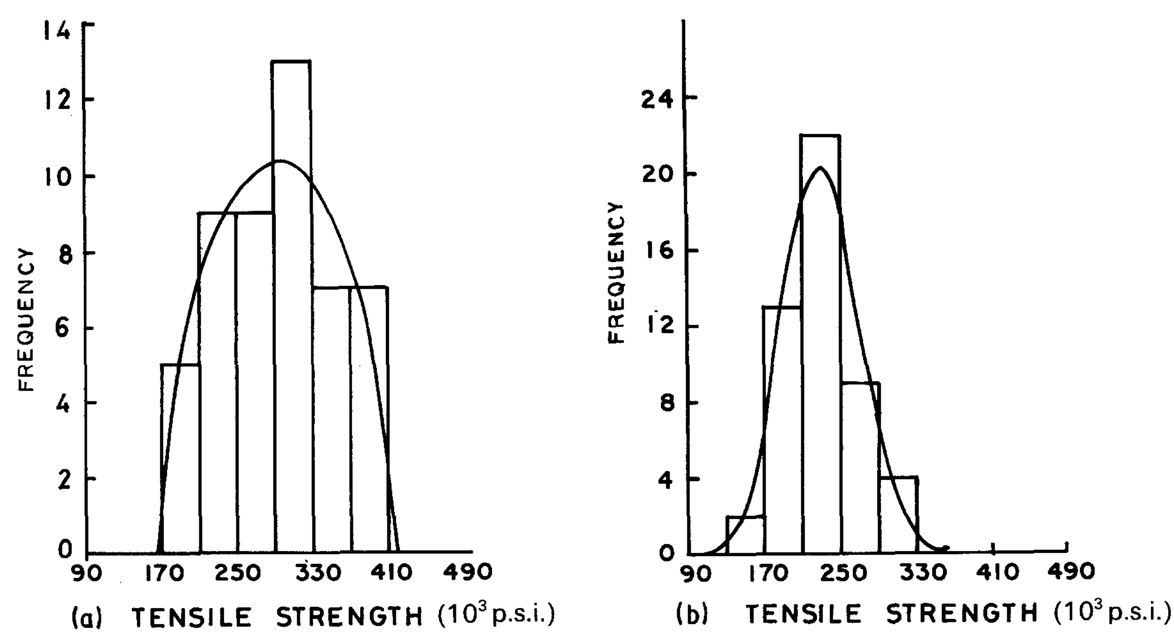

Figure 2 Tensile strength histograms ($N = 50$) of E-glass fibres having gauge lengths of (a) 15 mm and (b) 120 mm. The curves correspond to the fitted beta distribution. 10^3 p.s.i. = 6.895 MPa.

shown in Fig. 2. Snowden [16] has also analysed the statistical justification for using a bimodal Weibull distribution by calculating the standardized coefficients of skewness and kurtosis of various distributions, and concluded that the beta distribution fits the numerous bimodal strength data best.

In the beta distribution the values of the variates are limited to a finite interval, which is more realistic for the strength data of brittle materials like glass. It has also two shape parameters. On the other hand, the Weibull distribution requires $S = \infty$ for certainty of failure, which is physically an unsatisfactory boundary condition. To overcome this limitation Kies [8] proposed a modification to the Weibull distribution of the form

$$N(S) = \left(\frac{S - S_L}{S_u - S}\right)^{m_0} \quad (4)$$

where S_u is the upper strength limit and m_0 is defined as the damage coefficient. However, it has been shown [7, 12] that even the function given by Equation 4 is not applicable over the entire strength distribution. The functional form of Equation 4 is similar to one obtained from the beta distribution, except that it has only one shape parameter. Thus a further modification is suggested in the form

$$N(S) = \left(\frac{S - S_L}{S_{01}}\right)^{m_1} \Big/ \left(\frac{S_u - S}{S_{02}}\right)^{m_2} \quad (5)$$

where S_{01}, S_{02} and m_1, m_2 are scaling parameters and shape parameters, respectively. Equation 5 along with Equation 1 yields

$$\ln\left[\frac{1}{L}\ln\left(\frac{1}{1-P}\right)\right] = m_1 \ln\left(\frac{S - S_L}{S_{01}}\right) - m_2 \ln\left(\frac{S_u - S}{S_{02}}\right) \quad (6)$$

3. Average strength–length relationship

The average strength of the fibre can be written as

$$\bar{S} = \int_{S_L}^{S_u} P_S \, dS \quad (7)$$

where P_S = probability of survival = $(1 - P)$. Thus Equations 1 and 5 give

$$\bar{S} = S_L + \int_{S_L}^{S_u} \times \exp\left[-L\left(\frac{S - S_L}{S_{01}}\right)^{m_1} \Big/ \left(\frac{S_u - S}{S_{02}}\right)^{m_2}\right] dS \quad (8)$$

Thus, once the values of unknown parameters have been determined from Equation 6 by fitting the experimental data at one gauge length, the average strength values at any other gauge length can be determined from Equation 8 by numerical integration. However, as pointed out by Olshansky and Maurer [9] one disadvantage of determining the values of the unknown parameters of Equation 6 from the experimental data for exclusively one gauge length is that the majority of samples will fail over a limited range of stress levels, and little information of the flaw population outside this range will be obtained. This difficulty can be partially overcome by using data for at least two gauge lengths.

4. Data analysis

As mentioned earlier, the applicability of the proposed distribution function (Equation 5) has been evaluated in terms of the glass-fibre strength data reported by Schmitz and Metcalfe [7]. From the extensive strength data reported by them two sets of data have been analysed – one for E-glass fibres (Series II,1 in Table C-1 of [7]) and the other for S-994 glass fibres (9/19/62 in Table C-IX of [7]). 50 specimens were tested for E-glass fibres at each of the gauge lengths 15, 50, 60, 120 and 240 mm; for S-994 fibres, 25 specimens were tested at each of the gauge lengths 0.25, 0.5, 1, 2.5, 5, 7.5, 10, 15, 30 and 60 mm. For the details of experimental work reference may be made to their original work [7].

Fig. 1 shows the plot of $\ln\{(1/L)\ln[1/(1 - P)]\}$ against $\ln S$ for E-glass fibre tested at gauge lengths of 15 and 120 mm. Data for two gauge lengths differing by an order of magnitude have been used for the reasons mentioned earlier. Fig. 3 shows a similar plot for S-994 fibres for gauge lengths of 0.25 and 60 mm. Since in this case the fibre diameter shows a greater variation, the quantity L has been replaced by A where $A = \pi dL$, where d is the diameter of an individual fibre. In both cases the data points fall on a single curve.

For fitting Equation 6 to these data, initial estimates of S_u and S_L were made from the beta distribution fitted to the tensile strength histograms at short and long gauge lengths, respectively. A set of values were assumed for the parameters S_{01} and S_{02}, and the values m_1 and m_2 were determined by regression analysis of the data. From the calculated and experimental values of $\ln\{(1/L)\ln[1/(1 - P)]\}$, a least-squares sum was evaluated for the particular set of parameters S_{01} and S_{02}. The computation was then iterated with a new set of S_{01} and S_{02} until the minimum least-squares sum was obtained. The procedure was repeated by changing the values of S_u and S_L, again using the minimum least-squares sum as the criterion for the best fitted values. The values of parameters obtained in this way for both sets of data are given in Table I. The fitted equations are also plotted in Figs 1 and 3. The cumulative failure probabilities at different gauge lengths were evaluated from Equations 1 and 5. These are plotted in Figs 4 and 5 along with the experimental strength values of E and S-994 glass fibres, respectively. The

TABLE I Values of parameters of equation 6 obtained from failure probability analysis

Parameter	E-glass fibre	S-994 glass fibre
S_u (10^3 p.s.i. (GPa))	873.5 (6.02)	880.0 (6.07)
S_L (10^3 p.s.i. (GPa))	90.0 (0.62)	80.0 (0.55)
S_{01} (10^3 p.s.i. (GPa))	79.5 (0.55)	40.0 (0.28)
S_{02} (10^3 p.s.i. (GPa))	300.0 (2.07)	650.0 (4.48)
m_1	1.126	1.294
m_2	5.301	5.982

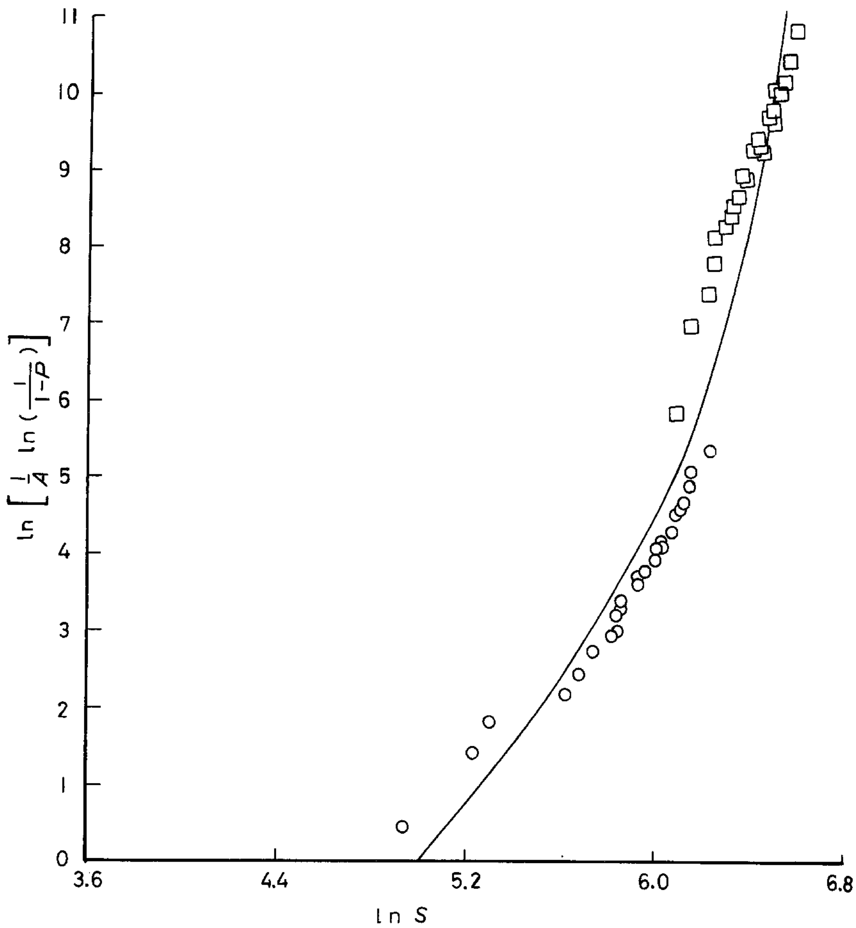

Figure 3 Plot of ln $\{(1/A) \ln [1/(1 - P)]\}$ against $\ln S$ for S-994 glass fibres at gauge lengths of (□) 0.25 and (○) 60 mm, from the Smitz and Metcalfe [7]. The curve corresponds to the fitted equation.

average strength values at different gauge lengths, obtained from Equation 8 by numerical integration, are given in Table II and plotted in Fig. 6 on a log–log scale for both E and S-994 glass fibres.

5. Discussion

Figs 1 and 3 show close agreement between the experimental data and the proposed distribution function. The values of the upper strength limit are obtained as 873.5 and 880 × 10^3 p.s.i. (6.023 and 6.068 GPa) for E and S-994 glass fibres, respectively. The limit for the tensile strength of a brittle, elastic solid can be derived from the force–displacement curve between two atoms. Accordingly, the theoretical maximum strength S_t is estimated to be [18]

$$S_t \simeq E/10$$

where E is the tensile modulus. Applying this equation, the theoretical upper limit for the strengths of E and S-994 glass fibres are obtained as 1050 and 1220 × 10^3 p.s.i. (7.240 and 8.412 GPa), respectively, from the elastic moduli values reported by Schmitz and Metcalfe [7]. The lower values, as obtained from the fitted equation, thus possibly indicate the presence of flaws even at the smallest gauge length. It may be mentioned that Schmitz and Metcalfe [7], on the basis

TABLE II Average strength values of E and S-994 glass fibres

Type of glass fibre	Gauge length (mm)	Measured values ± standard deviation (10^3 p.s.i.)*	Calculated values (10^3 p.s.i.)*
E	15.00	292 ± 57.8	301.8
	30.00	277 ± 40.1	272.2
	60.00	227 ± 50.6	246.0
	120.00	233 ± 39.8	223.1
	240.00	227 ± 45.3	203.2
S-994	0.25	597 ± 76.4	565.8
	0.50	579 ± 70.0	536.1
	1.00	573 ± 85.0	503.8
	2.50	526 ± 92.8	457.3
	5.00	533 ± 98.4	419.3
	7.50	404 ± 101.6	396.1
	10.00	463 ± 116.3	379.2
	15.00	452 ± 134.7	355.0
	30.00	408 ± 125.8	312.4
	60.00	364 ± 90.3	269.2

*10^3 p.s.i. = 6.895 MPa.

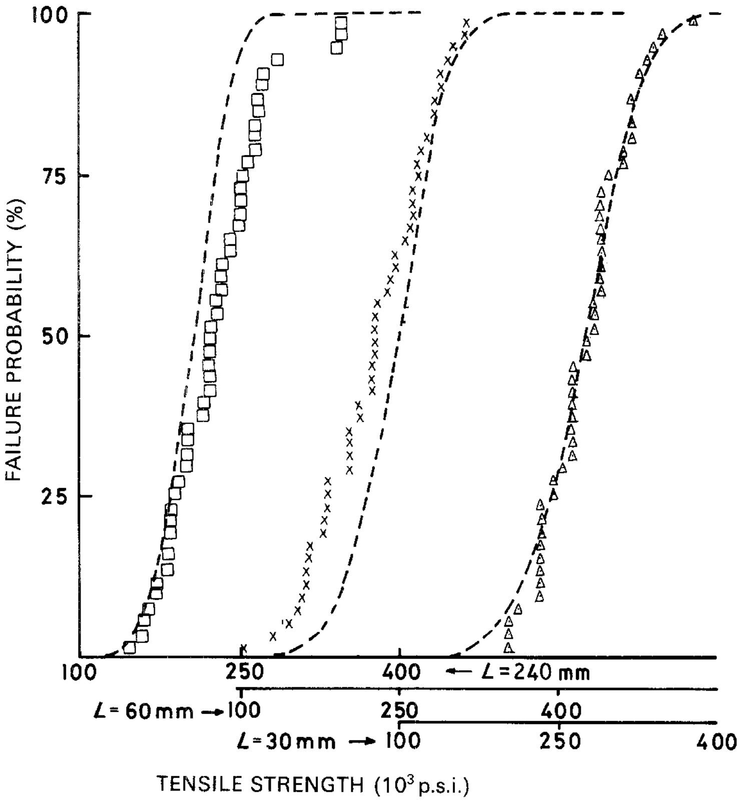

Figure 4 Failure probability of E-glass fibres having gauge lengths of (△) 30, (×) 60 and (□) 240 mm; (– – –) predicted. 10^3 p.s.i. = 6.895 MPa.

of the frequency of occurrence of high strength values, estimated a maximum strength of 800 × 10^3 p.s.i. (5.5 GPa) for S-994 fibres. This is in close agreement with the value obtained here. The values of S_L give the lower strength limit as the fibre length increases. The actual lowest measured strengths were 105 × 10^3 p.s.i. (724 MPa) for E-glass (at a probability of failure of 1%), and 94 × 10^3 p.s.i. (648 MPa) (at a probability of failure of 2%) for S-994 glass [7]. The values of 90 and 80 × 10^3 p.s.i. (621 and 552 MPa), obtained from the fitted equations for E and S-994 glass, respectively, are in close agreement with the above values.

As can be seen from Table I, the values of m_1 and m_2 obtained for both groups of fibre are of the same order of magnitude. Like the damage coefficient m_0 in Kies' equation (Equation 4), if these two coefficients are associated with the damage coefficients of fibre, the almost identical values of m_1 and m_2 for both E and S-994 glasses reflect the constancy of damaging factors in the manufacture and handling of glass fibre strands. This observation is in agreement with the conclusion drawn by Schmitz and Metcalfe [7] from the Weibull analysis.

Figs 4 and 5 show close agreement between the

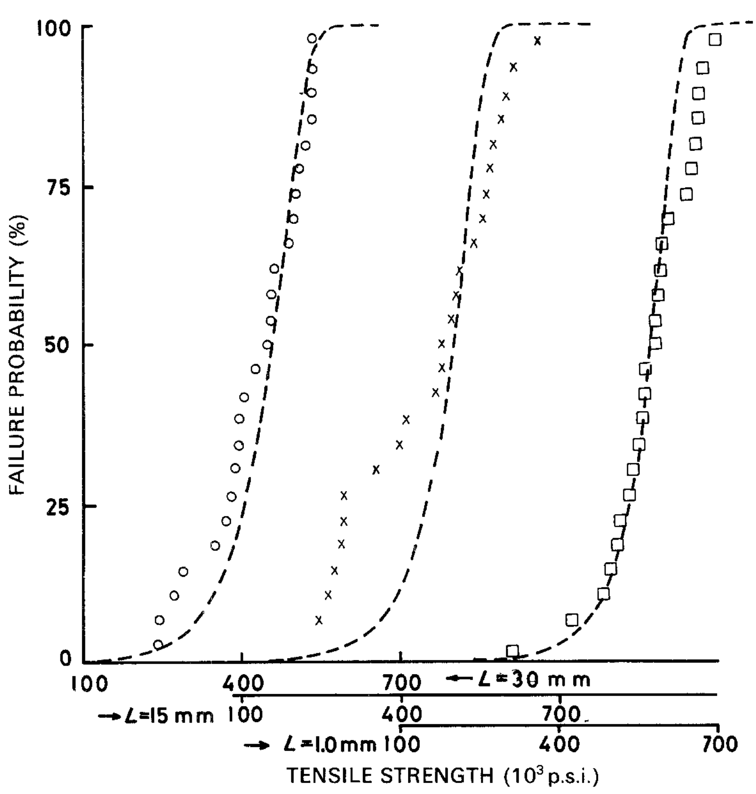

Figure 5 Failure probability of S-994 glass fibres having gauge lengths of (□) 1, (×) 15 and (○) 30 mm; (– – –) predicted. Data from Smitz and Metcalfe [7]. 10^3 p.s.i. = 6.895 MPa.

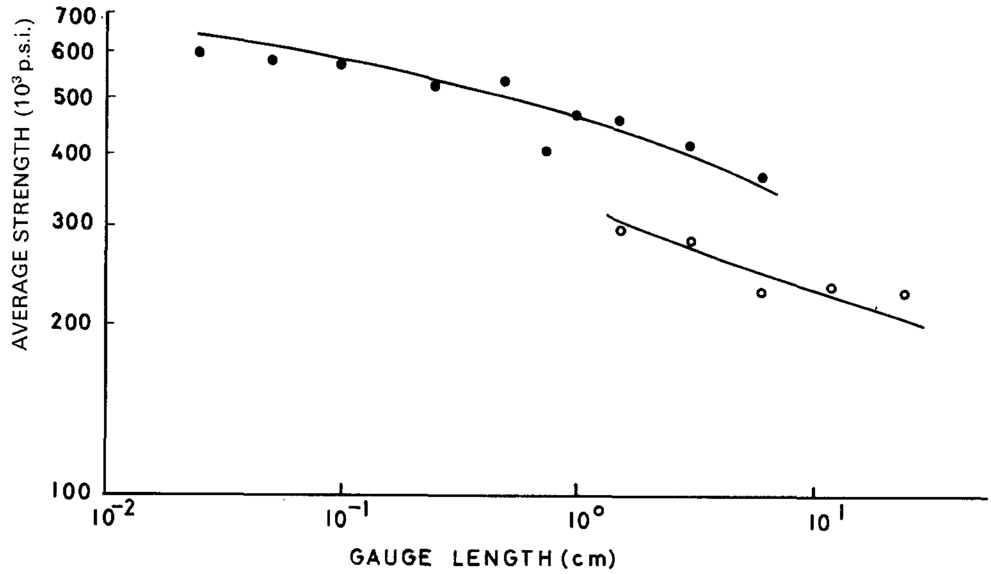

Figure 6 Log–log plot of average strength against gauge length for (○) E-glass and (●) S-994 glass fibres; (——) predicted. 10^3 p.s.i. = 6.895 MPa.

predicted failure probability and the experimental data at different gauge lengths, particularly at low gauge lengths. However, no ready explanation can be found for the large deviation of the lower group of strength values at a gauge length of 15 mm for S-994 glass from the experimental details. As a probable reason it can be attributed to non-randomness of the sample.

As can be seen from Table II and Fig. 6, the proposed equation predicts the gauge length dependence of strength quite accurately. In all cases the predicted value lies within the interval of one standard deviation from the mean. Fig. 6 shows that the rate of of increase of the slope of log(strength)–log(length) for S-994 glass increases at a gauge length of about 5 mm. This is consistent with the experimental observations of Schmitz and Metcalfe [7]. Though the strength data for E-glass fibre were reported only up to a gauge length of 15 mm, a similar trend is also predicted from the proposed equation.

6. Conclusions

The strength data of E and S-994 glass fibres have been analysed in terms of a proposed cumulative distribution function using the weakest-link theory. The strength–length relationship has also been derived. The specific conclusions that can be drawn from this study are:

1. The cumulative flaw distribution function given by

$$N(S) = \left(\frac{S - S_L}{S_{01}}\right)^{m_1} \bigg/ \left(\frac{S_u - S}{S_{02}}\right)^{m_2}$$

can be used to describe the strength distribution in glass fibre.

2. The function provides an upper and lower limiting strength. This is consistent with the boundary conditions of the physical phenomena it represents.

3. The parameters m_1 and m_2 can be associated with damage of the fibre during manufacture and handling, and are therefore independent of glass composition.

4. As opposed to the Weibull distribution, the equation correctly predicts an increase in slope of the log(strength)–log(length) curve with increase in gauge length.

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